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Lattice vertex algebras on general even, self-dual lattices

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Abstract: In this note we analyse the Lie algebras of physical states stemming from lattice constructions on general even, self-dual lattices $\Gamma^{p,q}$ with $p \geq q$. It is known that if the lattice is at most lorentzian, the resulting Lie algebra is of generalized Kac-Moody type (or has a quotient that is). We show that this is not true as soon as $q > 1$. By studying a certain sublattice in the case $q > 1$ we obtain results that lead to the conclusion that the resulting non-GKM Lie algebra cannot be described conveniently in terms of generators and relations and belongs to a new and qualitatively different class of Lie algebras.

Keywords: Conformal Field Models in String Theory, Conformal and W Symmetry.

Contents

1. Introduction

Vertex operators and their algebras have played an important rôle in the development of string theory [1, 2] and constructions of algebras in mathematics [3, 4, 5]. The most prominent examples have been chiral vertex algebras constructed from integral lattices which can be interpreted as quantized (bosonic) strings moving in toroidal spaces. Here the lattice is usually taken to be of euclidean or lorentzian signature corresponding to compact euclidean internal spaces or minkowskian spacetimes respectively and the rank of the lattice in this application is restricted by the no-ghost theorem [6].

The formalisation and use of vertex algebra techniques eventually led to the proof of the moonshine conjectures by Borcherds [4]. A major step on the road to this proof was the introduction of the notion of a generalized Kac-Moody algebra (GKM for short, see [7]). Algebras of this type can appear as Lie algebras derived from the vertex operator construction and have several nice properties, most notably the Weyl-Kac-Borcherds denominator formula

$$e^{\rho}\prod_{\alpha>0}(1-e^{\alpha})^{\mathrm{mult}(\alpha)} = \sum_{w\in W}\det(w)w(e^{\rho}\sum_{\alpha}\epsilon(\alpha)e^{\alpha})\,, \tag{1.1}$$

which served to provide many new product formulae for known automorphic forms [8, 9].

Generalized Kac-Moody algebras[1] have also been proposed to underly several structures in theoretical physics. They have been shown to control threshold corrections in $N = 2, d = 4$ compactifications of the heterotic string [10], count degeneracies of certain black holes [11], be relevant in Gromov-Witten theory [12] and possibly also as symmetries of M-theory [13, 14].

If the rank of the lattice is ≤ 26 and there is at most one time-dimension, the no-ghost theorem from string theory implies that the resulting Lie algebra is of GKM type or has a quotient that is. For rank > 26 arguments in [15] show that the algebra is still a GKM algebra.[2] Less is known about Lie algebras associated to lattices of signature (p, q) where both $p, q > 1$. They have been proposed to be of relevance for the study of algebras associated with BPS states in toroidal compactifications of heterotic string theories [16].

This note aims at providing some results about the Lie algebras obtained by lattice vertex operator constructions based on even, self-dual lattices and to see how and where they differ from GKM type algebras. As the properties of the root systems of the algebras are intimately related to the geometry of the lattices, we proceed by first studying systems of fundamental roots of these lattices and then consider the implications for the algebras.

Our basic results deal with the existence of Weyl-like vectors and suitable height functions for the lattice roots and we prove that neither exist. This makes the existence of a nice set of fundamental roots unlikely. We present some further evidence for this statement and conjecture that the even, self-dual lattices of signature (p, q) with both $p, q > 1$ do not possess a system of simple roots.

Similarly, the Lie algebra associated to the lattice at hand can then not be written in terms of generators and relations in a fashion analogous to the way one can construct GKM algebras. They thus must be part of a wider class of Lie algebras with vastly different characteristics.

The structure of this note is as follows. To be mostly self-contained, we first introduce some notation and terminology for the lattices (section 2), vertex algebras and GKM algebras (section 3). We indicate how the Lie algebra of physical states corresponding to the lattice is constructed. In section 4 we discuss basic properties of the lattice and algebra root systems and what we understand by a set of fundamental or simple roots. The main results mentioned above are contained in section 5. Finally, we offer some concluding remarks and possible further directions.

2. Lattices

By an integral lattice Λ of rank n we understand the free abelian group on a finite number of generators $\mathbf{e}_i$ $(i = 1, \ldots, n)$ together with a non-degenerate, symmetric bilinear form $(-,-) : \Lambda \times \Lambda \to \mathbb{Z}$ taking values in the integers, so that

$$\Lambda = \left\{ \sum_{i=1}^{n} n_i \mathbf{e}_i \,|\, n_i \in \mathbb{Z} \right\}. \tag{2.1}$$

[1] As the class of GKM algebras comprises the usual finite-dimensional and Kac-Moody algebras we restrict ourselves to discussing these.

[2] I am grateful to N. R. Scheithauer for bringing this point to my attention.

Each such lattice can be seen as a discrete set of points in the real vector space $\mathfrak{h} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ which inherits an inner product, also denoted $(-,-)$, from the bilinear form of the lattice. This inner product is not necessarily definite. We can associate to each integral lattice Λ a dual lattice Λ^* in $\mathfrak{h}$ by

$$\Lambda^* := \{\xi \in \mathfrak{h} \,|\, (\xi, \lambda) \in \mathbb{Z} \text{ for all } \ \lambda \in \Lambda\}\,. \tag{2.2}$$

For integral lattices we obviously have $\Lambda \subset \Lambda^*$. We call a lattice *even* if the norm squares of all its elements are even, i.e. $\lambda^2 = (\lambda, \lambda) \in 2\mathbb{Z}$ for all $\lambda \in \Lambda$. We call a lattice *self-dual* if it is identical to its dual lattice, i.e. $\Lambda = \Lambda^*$. Self-duality for integral lattices is equivalent to being *uni-modular*, i.e. the inner product matrix $g_{ij} = (\mathbf{e}_i, \mathbf{e}_j)$ has determinant of modulus one: $|\det(g_{ij})| = 1$. We say Λ has *signature* (p, q) if the inner product $(-,-)$ has p positive and q negative eigenvalues. Necessarily we have $p + q = n$ and we assume $p \geq q$ without loss of generality.

We will restrict our attention in this paper to even, self-dual lattices. It is a well-known result that such lattices exist if and only if $(p - q) \equiv 0 \mathrm{mod}\ 8$ and are unique if $q \geq 1$ [17]. These lattices are usually denoted by $\Gamma^{p,q}$.

If $q = 0$ we are dealing with euclidean even, self-dual lattices. There is one such lattice of rank 8 (the root lattice Γ^8 of the Lie algebra $\mathfrak{e}_8$), two such of rank 16 ($\Gamma^8 \oplus \Gamma^8$ and Γ^{16}, the root lattice of $\mathfrak{e}_8 \times \mathfrak{e}_8$ and half the weight lattice of $\mathfrak{so}(32)$ respectively). In 24 dimensions there are 24 even, self-dual lattices, the so-called Niemeier lattices among which the Leech lattice is distinguished by having no elements of norm squared equal to 2.

If $q = 1$ the lattices are said to be of lorentzian signature and sometimes also denoted by $\mathrm{II}_{n-1,1}$ with $n = 2, 10, 18, 26, \ldots$. The most prominent of these are $\mathrm{II}_{1,1}$ (the root lattice of the monster algebra [4]), $\mathrm{II}_{9,1}$ (the root lattice of the hyperbolic Kac-Moody algebra $\mathfrak{e}_{10}$ [18]) and $\mathrm{II}_{25,1}$ (the root lattice of the fake monster algebra [8]).

If $q > 1$ little seems to be known about these lattices in the literature and it is part of the aim of this note to establish a few basic properties of these lattices and their implications for the vertex algebras based on them.

3. Vertex algebras of lattices

We summarize the definition and the relevant properties of vertex algebras and their construction starting from integral lattice, including the associated Lie algebra of physical states. Relevant references are [1, 5, 19, 20].

3.1 Vertex algebras

A *vertex algebra with conformal element* is a quadruple $(\mathcal{F}, \mathcal{V}, \omega, \mathbf{1})$ consisting of a vector space $\mathcal{F}$, called space of states, a map $\mathcal{V} : \mathcal{F} \to (End\mathcal{F})[[z, z^{-1}]]$, and distinguished elements $\omega, \mathbf{1} \in \mathcal{F}$, termed the conformal vector and the vacuum respectively. The map $\mathcal{V}$ is to be interpreted as associating to each state $\psi \in \mathcal{F}$ a *vertex operator* $\mathcal{V}(\psi, z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n-1}$. One can think of a state ψ as being generated from the vacuum as $\psi = \lim_{z\to 0} \mathcal{V}(\psi, z)\mathbf{1}$. A vertex algebra has to satisfy

1. Regularity: $\psi_n\phi = 0$ for n sufficiently large
2. Vacuum: $\mathbf{1}_n\psi = \delta_{n+1,0}\psi$
3. Injectivity: $\psi_n = 0$ for all $n \in \mathbb{Z} \Leftrightarrow \psi = 0$
4. Conformal vector: $\omega_{n+1} = L_n$ and the L_n obey the Virasoro algebra and L_0 gives an integer grading of the space of states

$$\mathcal{F} = \oplus_{n\in\mathbb{Z}}\mathcal{F}_n$$

with

$$\mathcal{F}_n = \{\psi \in \mathcal{F}|L_0\psi = n\psi\}$$

5. Jacobi identity:

$$\sum_{i\geq 0}(-1)^i \binom{l}{i} \Big(\psi_{l+m-i}(\phi_{n+i}\xi) - (-1)^l\phi_{l+n-i}(\psi_{m+i}\xi)\Big) = \sum_{i\geq 0}\binom{m}{i}(\psi_{l+i}\phi)_{m+n-i}\xi\,. \tag{3.1}$$

The Jacobi identity contains the most information about the algebra. It comprises the ordinary Jacobi identity if evaluated for $l = m = n = 0$ and generalizations thereof.

A *vertex operator algebra* is a vertex algebra for which the spectrum of L_0 is bounded below and all $\mathcal{F}_n$ are finite-dimensional. These conditions have obvious physical motivations. Interesting non-trivial examples are given by local vertex operators in a conformal field theory as in the case of a bosonic string compactified on an even and self-dual lattice.

3.2 Lattice construction

For an integral lattice Λ as in section 2 the space $\mathfrak{h}/\Lambda$ is a torus and the vertex algebra we wish to associate to the lattice Λ is given by quantisation of the space of maps from the circle S^1 into this torus. More precisely, we consider the space

$$\hat{\mathfrak{h}} = \mathfrak{h}[z, z^{-1}] \oplus \mathbb{R}K \tag{3.2}$$

of Laurent polynomials in V with central extension. We write $\alpha_{n,i}$ for $\mathbf{e}_i[z^n]$ and $\xi(n) = \xi^i\alpha_{n,i}$ for a general monomial $\xi[z^n]$ expanded in these basis vectors when $\xi = \xi^i\mathbf{e}_i$. Define a Heisenberg Lie algebra structure on this space by

$$[\xi(m), \eta(n)] = m\delta_{m,-n}(\xi, \eta)K. \tag{3.3}$$

The space of states of the vertex algebra for the lattice Λ is

$$\mathcal{F}_\Lambda = S(\mathfrak{h}^-) \otimes \mathbb{R}\{\Lambda\}\,, \tag{3.4}$$

where $\mathfrak{h}^- = z^{-1}\mathfrak{h}[z^{-1}]$ are the negative mode oscillators, S denotes the symmetric algebra and $\mathbb{R}\{\Lambda\}$ is the twisted group algebra of the lattice. Twisted here refers to the introduction of a 2-cocycle $\varepsilon : \Lambda \times \Lambda \to \mathbb{Z}/2\mathbb{Z}$ which is necessary to obtain the right commutation relations, see [2] for details. Elements of the group algebra will be denoted by e^α with

$$e^\alpha e^\beta = e^{\alpha+\beta} \,. \tag{3.5}$$

The Sugawara construction supplies the Virasoro element by

$$\omega = \frac{1}{2}\sum_{i=1}^{n} h^i(-1)h_i(-1) \otimes e^0 \,, \tag{3.6}$$

where h^i, h_i are dual bases of $\mathfrak{h}$. The vacuum is given by $1 \otimes e^0$.

The map $\mathcal{V}$ associating a vertex operator to each state is defined in the usual way using the normal ordering of the Heisenberg modes. It can be shown that with these choices we have constructed a vertex algebra with conformal element [5].

3.3 Lie algebra of physical states

We can use the conformal structure of the lattice vertex algebra to define a Lie algebra structure on the space of physical states. Define

$$P_n := \{\psi \in \mathcal{F}_n : L_m\psi = 0 \text{ for } \ m > 0\} \tag{3.7}$$

to be the space of physical states with weight n. Then it can be shown [3] that the space $P_1/L_{-1}P_0$ carries the structure of a Lie algebra with respect to the product

$$[\psi, \phi] = \psi_0\phi = Res_{z=0}\mathcal{V}(\psi, z)\phi \,. \tag{3.8}$$

This construction works for all vertex algebras with conformal vector but we will only consider the case of vertex algebras derived from a lattice construction for even, self-dual lattices $\Gamma^{p,q}$. In this case the root lattice of the Lie algebra of physical states is just $\Gamma^{p,q}$ and the roots are the non-zero elements with norm squared less than 2 and the multiplicity of a root $\alpha \in \Gamma^{p,q}$ is given by the classical partition function

$$\mathrm{mult}(\alpha) = p_{d-1}\left(1 - \frac{1}{2}\alpha^2\right) - p_{d-1}\left(-\frac{1}{2}\alpha^2\right), \tag{3.9}$$

where the two terms describe the number of independent elements for given norm squared in P_1 and P_0. $p_k(M)$ is the number of partitions of an integer M into positive integers with k colours and is generated by

$$\sum_{m\geq 0} p_k(m)q^m = \prod_{m\geq 1}(1-q^m)^{-k} \,. \tag{3.10}$$

The Cartan subalgebra is given by all states with zero momentum and arbitrary polarization vector in $\Gamma^{p,q} \otimes \mathbb{R}$.

If this construction is applied to the lattice $\mathrm{II}_{25,1}$ one arrives at the fake monster algebra [8] which is an example of a generalized Kac-Moody algebra (which will be defined in the following section 3.4) and in general for a lorentzian lattice we obtain a GKM algebra [15]. So the Lie algebras for $\Gamma^{p,q}$ with $q \leq 1$ and are at most of generalized Kac-Moody type, often they are even finite-dimensional or of Kac-Moody type.

3.4 Generalized Kac-Moody algebras

Following [7], we briefly review a few standard facts about generalized Kac-Moody algebras. A *generalized Kac-Moody algebra* is defined via a symmetrized Cartan matrix $A = (a_{ij})$, with i, j in a possibly infinite but countable index set I, satisfying the following conditions

$$\begin{aligned} a_{ij} &= a_{ji}, \\ a_{ij} &\leq 0 \text{ if } i \neq j, \\ \frac{2a_{ij}}{a_{ii}} &\in \mathbb{Z} \text{ if } a_{ii} > 0\,. \end{aligned} \tag{3.11}$$

We also assume the existence of a real vector space H with symmetric bilinear inner product (not necessarily positive definite) and elements $h_i \in H$ $(i \in I)$ such that $(h_i, h_j) = a_{ij}$. Then we define the generalized Kac-Moody algebra G to be the Lie algebra generated by H and e_i and f_i subject to the relations

1. The image of H in G is commutative.
2. $h \in H$ acts diagonally on the e_i, f_i: $[h, e_i] = (h, h_i)e_i$ and $[h, f_i] = -(h, h_i)f_i$.
3. $[e_i, f_i] = h_i$, $[e_i, f_j] = 0$ if $i \neq j$.
4. If $a_{ii} > 0$ then $(\mathrm{ad}\, e_i)^{1-2a_{ij}/a_{ii}} e_j = 0$ and $(\mathrm{ad}\, f_i)^{1-2a_{ij}/a_{ii}} f_j = 0$ (Serre relations).
5. If $a_{ij} = 0$ then $[e_i, e_j] = [f_i, f_j] = 0$.

The main difference compared to finite-dimensional simple Lie algebras or Kac-Moody algebras is that imaginary simple roots are permitted as a_{ii} can be less or equal to 0. The multiplicity of an imaginary simple root can be greater than 1. Also, the number of simple roots can be infinite. Most terminology and properties carry over. We extend the bilinear form on H to the root lattice Q which is the free abelian group on the simple elements. The Weyl group is generated by reflections in the real simple roots, and roots can be grouped into positive or negative ones according to whether they are a sum of simple roots or the negative thereof.

Among the standard properties of GKM algebras we mention the following. The denominator formula (1.1) involving the Weyl vector determined by the real simple roots holds. Unless for extremely degenerate cases, a GKM algebra G has an integer grading $G = \oplus G_m$ such that each G_m is finite-dimensional if $m \neq 0$ and G_0 is abelian.[3] Similar to Kac-Moody algebras, G has an invariant bilinear form such that G_α and G_β are orthogonal unless $\alpha + \beta = 0$, and the hermitean form constructed from this by use of the Chevalley involution is positive definite on the root spaces.

4. Reflections, algebra and lattice roots

In order to analyse the Lie algebra of physical states we construct from lattices further we need to understand the root system and the geometry of the root lattice. In this section we introduce the required notation.

[3]This is true unless the Cartan matrix has a infinite number of identical rows.

4.1 Lattice roots and algebra roots

Following [21], we call an element $\alpha \in \Gamma^{p,q}$ a *lattice root* if the associated elementary reflection

$$w_\alpha(\gamma) = \gamma - 2\frac{(\gamma,\alpha)}{(\alpha,\alpha)}\alpha, \quad \gamma \in \Gamma^{p,q} \tag{4.1}$$

is a symmetry of the lattice. It is not difficult to see that for even, self-dual lattices the lattice roots are precisely the elements of norm squared 2:

$$\Delta^{\text{latt}} = \left\{\alpha \in \Gamma^{p,q} \mid \alpha^2 = 2\right\}. \tag{4.2}$$

The set of all reflection forms a group $\mathcal{R}$, called the reflection subgroup of the automorphism group of the lattice.

If $\mathfrak{g}$ is a Lie algebra with root lattice $\Gamma^{p,q}$ and Cartan subalgebra $\mathfrak{h}$ then it possesses a natural grading

$$\mathfrak{g} = \bigoplus_{\alpha \in \Gamma^{p,q}} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \bigoplus_{0 \neq \alpha \in \Gamma^{p,q}} \mathfrak{g}_\alpha\,, \tag{4.3}$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}\,. \tag{4.4}$$

If $\mathfrak{g}_\alpha \neq \{0\}$ and $\alpha \neq 0$ then α is called an *algebra root* or root for short and $\mathfrak{g}_\alpha$ the associated root space. If $\alpha^2 > 0$, α is called *real* and *imaginary* otherwise, denoted by Δ^{re} and Δ^{im} respectively. For the lattice Lie algebras we are interested in all real roots have norm squared 2. In fact, here all α with $\alpha^2 = 2$ are algebra roots and

$$\Delta^{\text{latt}} = \Delta^{re} \tag{4.5}$$

for these lattice Lie algebras. This is not true in general, for instance for most Kac-Moody algebras. In a similar fashion to above, the real roots have reflection symmetries which form a group $\mathcal{W}$ called the Weyl group of the algebra. We note that the set of all roots is invariant under $\alpha \to -\alpha$, also reflected in the Chevalley involution of the Lie algebra.

4.2 Fundamental and simple roots

It is an important question if the set of all roots can be conveniently described in terms of some generators as for GKM type algebras. As upon commutation $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ an element which can be written as the sum (or commutator) of other elements cannot be part of a minimal generating set. It is natural to look for such a generating set by a minimality condition with respect to addition. For this it is first necessary to define the notion of positivity and negativity of a (lattice or algebra) root which respects the group operation on the lattice. This can be done, for instance, by finding a hyperplane in the vector space underlying the lattice not containing any of the roots (real or imaginary). Then call one side of the hyperplane positive, the other one negative.

For the case of the lattice roots, those positive roots which cannot be written as a sum of other positive roots will be called *fundamental roots* and are denoted (α_i) for i in some index set I.[4] The set of fundamental roots is sufficient to describe the reflection group $\mathcal{R}$ and the information can be encoded in a Dynkin diagram or a Cartan-like matrix.

For the lattice Lie algebra, those positive roots whose root spaces contain elements which cannot be written as a commutator of elements of the root spaces of other positive roots will be called *simple roots* with multiplicity corresponding to the number of linearly independent such elements in their root spaces. The fundamental roots of the lattice will turn into simple real roots of the algebra and then in principle one might try to determine the additional simple roots recursively if one has a suitable grading like the distance to the hyperplane chosen for splitting the roots into positive and negative. Again, the simple roots will be denoted α_i $(i \in I)$ and their inner-product matrix contains all information about the Lie algebra.

It seems an open question under which circumstances such a generating set exists and can actually be determined.

4.3 Weyl-like elements

An important quantity for generalized Kac-Moody algebras is the Weyl element ρ which is defined by

$$(\rho, \alpha_i) = 1 \tag{4.6}$$

for all real simple roots α_i where one might need to extend $\mathfrak{h}$ by derivations if the Cartan matrix is singular. It derives its importance from the rôle it plays in the character formula and, in light of the discussion in the preceding paragraph, can be seen to be an optimal choice of grading for the positivity problem.[5] This said, we call an element $\rho \in \Gamma^{p,q}$ *Weyl-like* if it has non-vanishing inner product with all lattice roots:

$$(\rho, \alpha) \neq 0 \ , \text{ for all } \alpha \in \Delta^{\text{latt}} \, . \tag{4.7}$$

Such an element can then be used to find the fundamental roots of the lattice.[6]

5. Properties of the Lie algebra of physical states based on the lattice $\Gamma^{p,q}$ for $q > 1$

Whereas the Lie algebra $\mathfrak{g}$ of physical states for at most lorentzian lattices can be classified as generalized Kac-Moody algebras (GKM), we will see that the situation is different as soon as $q > 1$.

[4]This notion is not fully equivalent to saying that the fundamental roots are orthogonal to the faces of a fundamental domain for the action of $\mathcal{R}$ on $\mathfrak{h}$. The image of the fundamental chamber $\cap_{i\in I} \{\xi \in \mathfrak{h} \mid (\alpha_i, \xi) \geq 0\}$ under $\mathcal{R}$ is the so-called Tits cone X. An example when $X \neq \mathfrak{h}$ is given by $\mathrm{II}_{9,1}$. Nevertheless, X contains all lattice roots.

[5]For this we might require that the lattice roots span the lattice.

[6]For most purposes one might also require that to each integer m there are only finitely many roots which have inner product m with ρ. That this is not strictly necessary can be seen from the example of $\mathrm{II}_{25,1}$.

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The lattice $\Gamma^{p,q}$ can be written as

$$\Gamma^{p,q} = \Gamma^{8s} \oplus \underbrace{\mathrm{II}_{1,1} \oplus \cdots \oplus \mathrm{II}_{1,1}}_{q}\,, \tag{5.1}$$

where Γ^{8s} is a euclidean lattice of rank $8s$ where $s = (p-q)/8$ and there are q summands of the basic light-cone lattice $\mathrm{II}_{1,1}$. For $q > 1$, $\Gamma^{p,q}$ contains at least one copy of $\Gamma^{2,2} = \mathrm{II}_{1,1} \oplus \mathrm{II}_{1,1}$ as a sublattice. Many of the problems in analysing $\mathfrak{g}$ based on $\Gamma^{p,q}$ can already be understood by studying $\Gamma^{2,2}$.

A convenient description of the lattice $\Gamma^{2,2}$ is given in terms of a double set of light-cone co-ordinates:

$$\Gamma^{2,2} = \{(k,l;m,n) \mid k,l,m,n \in \mathbb{Z}\}\,, \tag{5.2}$$

equipped with the metric

$$|(k,l;m,n)|^2 = -2kl - 2mn\,. \tag{5.3}$$

As discussed above. the lattice roots are exactly those elements of $\Gamma^{2,2}$ with $\alpha^2 = 2$ or, put differently, the integral solutions to the non-linear diophantine problem

$$kl + mn = -1\,. \tag{5.4}$$

There is an obvious one-to-one correspondence between the lattice roots and elements of $M \in \mathrm{SL}(2,\mathbb{Z})$ by making use of the bijection $k \leftrightarrow -k$ and multiplying (5.4) by -1.

Now we are able to show some interesting properties of the lattice.

Proposition 1. *There is no Weyl-like element for the lattice* $\Gamma^{2,2}$.

Proof. Assuming that there is such an element $\rho = (a,b;c,d)$ which can be written as a 2×2-matrix with integer entries

$$A = \begin{pmatrix} -b & c \\ d & a \end{pmatrix}, \tag{5.5}$$

we have to show that there always exists an element $R \in \mathrm{SL}(2,\mathbb{Z})$

$$R = \begin{pmatrix} -k & m \\ n & l \end{pmatrix} \tag{5.6}$$

(corresponding to a lattice root $\alpha = (k,l;m,n)$) such that

$$\mathrm{tr}(RA) = al + bk + cn + dm = -(\rho,\alpha) = 0\,. \tag{5.7}$$

The condition $\mathrm{tr}(RA) = 0$ for some $R \in \mathrm{SL}(2,\mathbb{Z})$ is equivalent to the condition

$$\mathrm{tr}(PAQ) = 0\,, \tag{5.8}$$

for some matrices $P, Q \in \mathrm{SL}(2,\mathbb{Z})$. Hence, if we can bring A to a traceless form by elementary row and column manipulations we are done. But by a variant of Euclid's algorithm we can always arrive at the following form for A by doing such manipulations

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_1 e \end{pmatrix}, \tag{5.9}$$

for some $d_1 \neq 0$ (unless $A = 0$ to start with) and $e \in \mathbb{Z}$. Now, we can add the first column to the second and then subtract $e+1$ times the first row off the second and the resulting matrix will be traceless. □

The significance of proposition 1 is that we cannot find a linear "height" function $h : \Gamma^{2,2} \to \mathbb{Z}$ which is non-vanishing on the lattice roots and can be used to distinguish positive and negative roots and to subsequently determine the fundamental roots. Conversely, supposing we have somehow arrived at a set of fundamental roots, so that all lattice roots can be written either as a sum or minus a sum of fundamental roots, there cannot be a Weyl vector for this set of fundamental roots in $\Gamma^{2,2}$. Proposition 1 shows that $\Gamma^{2,2}$ cannot be graded by integers in a fashion desirable for GKM algebras.

Proposition 2. *Every group homomorphism $h : \Gamma^{2,2} \to \mathbb{R}$ which is non-vanishing on Δ^{latt} has zero as an accumulation point of its image.*

Proof. By proposition 1 we only need to consider the case when h takes irrational values, at least for some lattice roots. Let $(k_0, l; m_0, n)$ be such an element. We know that l, n are coprime and that for all $t \in \mathbb{Z}$

$$(k_0 + tn, l; m_0 - tl, n) \tag{5.10}$$

is also a root.[7] If h is given by

$$h(k, l; m, n) = -al - bk - cn - dn\,, \tag{5.11}$$

let

$$\begin{aligned} h_{l,n}(t) &= -al - b(k_0 + tn) - cn - d(m_0 - tl) = (dl - bn)t - al - bk_0 - cn - dm_0 \\ &= Mt + N\,. \end{aligned} \tag{5.12}$$

There exists a $t_0 \in \mathbb{Z}$ such that $h_{l,n}(t_0) \in (0, M)$. Now approximate b and d over the rationals as

$$b \approx \frac{p_1}{q_1},\ d \approx \frac{p_2}{q_2} \tag{5.13}$$

with $q_1 p_2$ and $q_2 p_1$ coprime. Choosing $l = q_2 p_1$ and $n = q_1 p_2$ gives $M = dl - bn \approx 0$. By increasing the accuracy of the approximation we can make M arbitrarily small, showing that 0 is an accumulation point of $h(\Delta^{\mathrm{latt}})$. □

Thus, even though it is possible to split the lattice roots into positive and negative roots by a general hyperplane there is never a minimal element that could be identified as fundamental. Combining proposition 1 and proposition 2, we get

[7] Actually, all solutions to the lattice root condition are of this form: For every coprime pair l, n there is such a one-parameter family of solutions.

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Corollary 1. *Let $H \subset \Gamma^{2,2} \otimes \mathbb{R}$ be a hyperplane. Then either H contains an element of Δ^{latt} or there is an element of Δ^{latt} arbitrarily close nearby.*

Interpreting the possible maps h as points in $\Gamma^{2,2} \otimes \mathbb{R}$ we also get that to each point in $\Gamma^{2,2} \otimes \mathbb{R}$ there is a hyperplane orthogonal to a lattice root passing arbitrarily close by. This has implications for the reflection group $\mathcal{R}$.

Corollary 2. *Any potential fundamental domain of the group $\mathcal{R}$ generated by reflections in the lattice roots of $\Gamma^{2,2}$ acting on $\Gamma^{2,2} \otimes \mathbb{R}$ has empty interior.*

Proof. Splitting $\Delta^{\text{latt}} = \Delta^{\text{latt}}_{+} \cup \Delta^{\text{latt}}_{-}$ every single reflection in an element $\alpha \in \Delta^{\text{latt}}_{+}$ exchanges the two half spaces

$$\begin{aligned} H_\alpha^+ &= \left\{\gamma \in \Gamma^{2,2} \otimes \mathbb{R} |\ (\alpha,\gamma) \geq 0\right\} \quad \text{and} \\ H_\alpha^- &= \left\{\gamma \in \Gamma^{2,2} \otimes \mathbb{R} |\ (\alpha,\gamma) \leq 0\right\}, \end{aligned} \tag{5.14}$$

leaving the orthogonal elements invariant. We pick the positive half space and try to construct a fundamental domain D for the action of $\mathcal{R}$ on $\Gamma^{2,2} \otimes \mathbb{R}$.

$$D = \bigcap_{\alpha \in \Delta^{\text{latt}}_{+}} H_\alpha^+ \tag{5.15}$$

is a closed set with the property that every reflection takes a point in the interior of D to a point outside of D and D is bounded by hyperplanes orthogonal to the lattice roots. By corollary 1, any point in $\Gamma^{2,2} \otimes \mathbb{R}$ has such a hyperplane nearby and thus the interior of D has to be empty. □

This also relates to the absence of a Weyl vector which would lie in the interior of such a set and a "ball" of radius 1 around its tip touching all the faces. We can describe D more explicitly when we use a height function h as above. Then D is a ray in $\Gamma^{2,2} \otimes \mathbb{R}$, precisely the one in the direction of the point determining h and the Tits cone is $(\Gamma^{2,2} \otimes \mathbb{R}) \backslash \Gamma^{2,2}$.

We have accumulated several pieces of evidence for

Conjecture 1. *The lattice $\Gamma^{2,2}$ does not possess a system of fundamental roots.*

We remark that choosing non-linear orderings of the lattice roots like lexicographic ordering one can find a few fundamental elements which can be shown not be a generating set of all roots. Completing them to a set of fundamental roots cannot be done algorithmically as there is no analogue of height, i.e. a superlinear functional that ensures that one only has to consider a given set of roots when asking if a particular root can be written as a sum of "lower" roots. As the fundamental roots of the lattice carry over by construction to the real simple roots of the Lie algebra of physical states $\mathfrak{g}$. As $\Gamma^{2,2} \subset \Gamma^{p,q}$ for $p \geq q > 1$ we are led to the believe that $\mathfrak{g}$ cannot have a system of simple roots analogous to the construction of generalized Kac-Moody algebras.

Using the results on the lattice $\Gamma^{2,2}$ we can now prove

Proposition 3. *The Lie algebra of physical states $\mathfrak{g} = P_1/L_{(-1)}P_0$ associated to the lattice $\Gamma^{p,q}$ is not of generalized Kac-Moody type for $q > 1$.*

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Proof. We restrict to the subalgebra where we only consider the lattice $\Gamma^{2,2}$ and assume that $\mathfrak{g}$ is a GKM algebra and show that this leads to a contradiction. So suppose $\mathfrak{g}$ is generated by a set of simple roots $\{\alpha_i : i \in I \subset \mathbb{N}\} = \Pi = \Pi^{re} \uplus \Pi^{im}$ with generalized Cartan matrix $a_{ij} = (\alpha_i, \alpha_j)$ and we can exclude zero rows. We denote the set of roots of $\mathfrak{g}$ as $\Delta^Q \subset Q$ which splits into positive ($\alpha > 0$) and negative ($\alpha < 0$) roots $\Delta^Q = \Delta^Q_+ \uplus \Delta^Q_- \subset Q_+ \uplus Q_-$. We define the fundamental set $C = \{\alpha \in Q_+ | (\alpha, \alpha_i) \leq 0$ for all real simple α_i and the support of α is connected$\} \setminus \bigcup_{j \geq 2} j\Pi^{im}$.

We know that our algebra is graded by the lattice $\Gamma^{2,2}$ as $\mathfrak{g} = \bigoplus_{\alpha \in \Gamma^{2,2}} \mathfrak{g}_\alpha$, where each $\mathfrak{g}_\alpha$ is finite dimensional and we are abusing notation by denoting roots in Q and $\Gamma^{2,2}$ by α. If $\mathfrak{g}$ is a GKM algebra then $\Gamma^{2,2} \otimes \mathbb{R}$ will be the quotient of $Q \otimes \mathbb{R}$ by the kernel K of the Cartan matrix. Suppose the kernel contains a (positive) root α of $\mathfrak{g}$ then the norm of this root is $\alpha^2 = 0$ and thus α is an isotropic root in the Weyl chamber. As we excluded zero rows the support of α is affine. For a simple root β from the support of α, the elements $\beta + n\alpha$ would be roots for all natural n, giving rise to infinite multiplicities in the quotient contradicting what we know about $\mathfrak{g}$. Hence K does not contain any root.

Now suppose there are elements $\alpha > 0$ and $\beta < 0$ which differ by an element κ of the kernel K, i.e. $\alpha = \beta + \kappa$. This κ obviously has to be positive and lies in the Weyl chamber. As it is not a root it has to have disconnected support in order not to lie in the fundamental set. So it is the sum of null roots of disconnected subdiagrams of the Dynkin diagram of $\mathfrak{g}$. But then each of the summands is also in the kernel (by disconnectedness) and thus a null root of $\mathfrak{g}$ which contradicts the fact above that there are no roots in the kernel. Hence $K \cap Q_\pm = K \cap \Delta^Q_\pm = \{0\}$. This means that the quotient map takes $Q_\pm$ to disjoint, convex sets $\Gamma^{2,2}_\pm$ which by linearity are exchanged by $p \leftrightarrow -p$ and thus can be split by a hyperplane in $\Gamma^{2,2}$. By the properties of $\Gamma^{2,2}$ such a hyperplane must have a normal vector ρ with irrational coordinates and the fundamental domain is the ray in the direction of ρ. This is where all imaginary simple roots must live but as the coordinates of ρ are irrational this ray does not contain any lattice points and so there are no imaginary simple roots for this Lie algebra. So $\mathfrak{g}$ can be at most a standard Kac-Moody algebra [22] with an infinite number of (real) simple roots.

The set Δ^{im}_+ (Δ^{im}_-) is Weyl invariant so that any sum of positive (negative) imaginary roots roots will again be an imaginary root if it is a root. If we separate the positive and negative roots in $\Gamma^{2,2}$ by $\rho = (a, b; c, d)$, we can assume that $b \neq 0$ without loss of generality as we can swap a and b by a Weyl reflection and not both can vanish. Now consider the roots $\beta_1 = (N, 0; 1, 0)$ and $\beta_2 = (N, 0; 0, -1)$ which are both imaginary and if we choose N large enough both are either positive or negative. Their sum is then also positive or negative but a real root contradicting the structure on $\Delta^{im}_\pm$ explained above. □

Hence, $\mathfrak{g}$ is an element of a wider class of Lie algebras. As the defining relations of generalized Kac-Moody algebras are automatically satisfied due to the vertex operator construction, we observe that if there were simple roots for $\mathfrak{g}$ they they would need to violate at least one of the basic properties of generalized Cartan matrices (3.11). Closer inspection reveals that their inner product matrix would need to have positive off-diagonal entries. For such Lie algebras we can work out the multiplicities for any given root inductively by height using the theory of generalized Kac-Moody algebras.

Theorem 1. *For $q > 1$, the Lie algebra $\mathfrak{g}$ associated to $\Gamma^{p,q}$ cannot be described in terms of simple generators and relations analogous to GKM algebras, say, encoded in Cartan matrices or Dynkin-like diagrams.*

Proof. Assume there is a system of generators for $\mathfrak{g}$ and as argued above they satisfy by construction relations similar to the ones of GKM algebras. But their Cartan matrix cannot be that of a GKM algebra as we have shown that this leads to a contradiction. So we have to relax the property that we only allow non-positive off-diagonal elements in the Cartan matrix. But for the elements used above we can view these as elements of an embedded GKM algebra (if we restrict the height accordingly) where we can still obtain the same contradiction as in the proof of proposition (3). □

6. Conclusions and open questions

The limitations on the type of Lie algebra one can obtain for this general lattice construction seem rather tight. Is there any other way to describe these algebras conveniently except for saying that they derive from the lattice construction? For instance, the class of algebras excluded here rest on amalgamating $\mathfrak{sl}_2$ subalgebras — maybe if one took other basic objects one can construct these Lie algebras. Is there some generalization of a character or denominator formula one could write down? As there is no Weyl-like vector and also the Weyl group seems rather unwieldy, it is not clear how to suitably replace the quantities appearing in the denominator formula (1.1). The representation theory for these algebras should also prove interesting.

Besides these mathematical questions it is not clear what the relevance of these algebras could be in theoretical physics, along the lines of [10, 16]. If algebras of BPS states turn out to be similar to these general lattice Lie algebras it is essential to get some handle on the underlying structure.

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